Introduction to Monte Carlo methods

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$\Rrightarrow$

https://github.com/mchrzasz/EMPP_MC

$\Rrightarrow$

$\rightarrowtail$

$\Rrightarrow$

$F \hat{F}$

$$\hat{F} = f(\{r_1, r_2, r_3, ..., r_n\}; ...),$$

$\{r_1, r_2, r_3, ..., r_n\}$

$\twoheadrightarrow$

-
- π
-
-
- FERMIAC
-

Lecture1/Eulernumber

$\Rrightarrow$

$\rightarrowtail ee = 2.7182818 ... \Rrightarrow \hat{e}$

- $(0,1)\mathcal{U}(0,1)$

$$x_1 < x_2 < ... < x_{n-1} > x_n$$

- $nNne$

$$\hat{e} = \frac{1}{N}\sum_{i=1}^{N} n_i \xrightarrow{N\to\infty} e.$$

	N	$\hat{e}$	$\hat{e} - e$	
		2.760000	0.041718	
$\Rrightarrow$		2.725000	0.006718	$\sim \sqrt{N}$
		2.718891	0.000609	
		2.718328	0.000046	

$\sqrt{N}$Lecture1/Eulernumber

$\Rrightarrow$

$\rightarrowtail\sqrt{N}$

$\rightarrowtail$

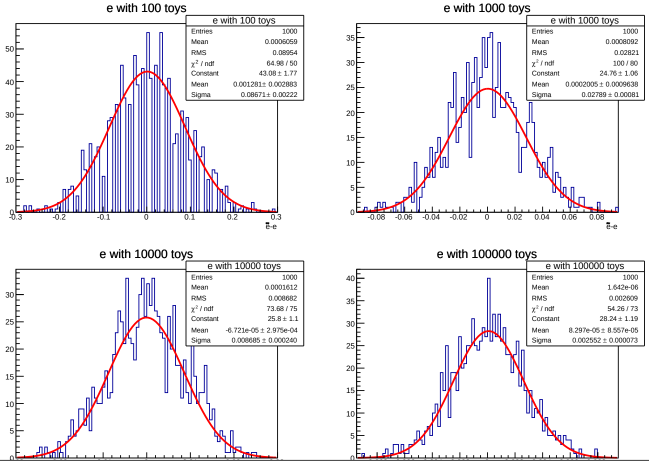
$\rightarrowtail$

$N N \in (100, 1000, 10000, 100000) n_N \hat{e} - e N$

$\sqrt{N}$ Lecture1/Eulernumber

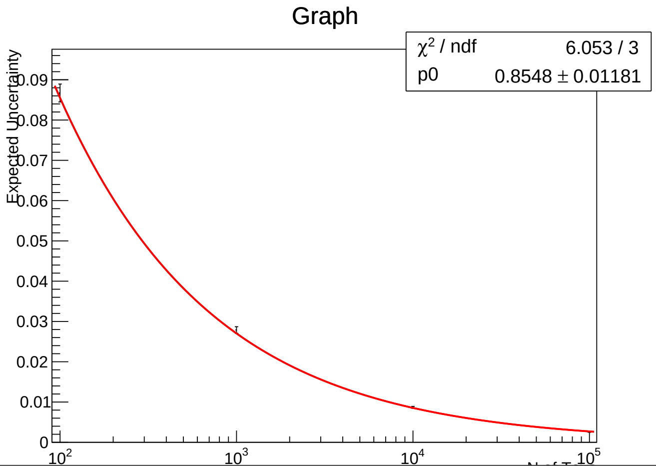
$\rightarrowtail$

$\rightarrowtail$



$\sqrt{N}$ Lecture1/Eulernumber

- $\rightarrowtail$
- $\rightarrowtail$



$\hookrightarrow$

$\Rrightarrow r_i \mathcal{U}(0,1)$

$$F = F(r_1, r_2, ... r_n)$$

$$I = \int_0^1 ... \int_0^1 F(x_1, x_2, ..., x_n) dx_1, dx_2 ..., dx_n$$

I

$$E(F) = I.$$

$\Rrightarrow$

$(0,1)$

$$\frac{1}{N} \sum_{i=1}^{N} f(x_i) \xrightarrow{N \to \infty} E(f) = \frac{1}{b-a} \int_a^b f(x) dx$$

$\Rrightarrow N \to \infty$

$\longmapsto$

$\rightarrowtail$

$$V(\hat{I}) = \frac{1}{n}\left\{E(f^2) - E^2(f)\right\} = \frac{1}{n}\left\{\frac{1}{b-a}\int_a^b f^2(x)dx - I^2\right\}$$

$\hookrightarrow V(\hat{I}) I$

$\Rrightarrow V(\hat{I})$

$$\hat{V}(\hat{I}) = \frac{1}{n}\hat{V}(f), \qquad \hat{V}(f) = \frac{1}{n-1}\sum_{i=1}^{n}\left[f(x_i) - \frac{1}{n}\sum_{i=1}^{n} f(x_i)\right]^2.$$

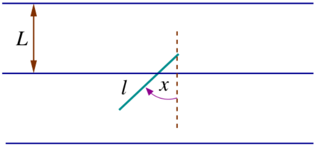
$\Rrightarrow \hat{\sigma} = \sqrt{\hat{V}(\hat{I})}$

π

$\Rightarrow l L \pi$

$\Rightarrow x \in \mathcal{U}(0,\pi) \Rightarrow$

$$\rho(x) = \frac{1}{\pi}$$



$\Rightarrow p(x)$

$$p(x) = \frac{l}{L}|\cos x|$$

n

N

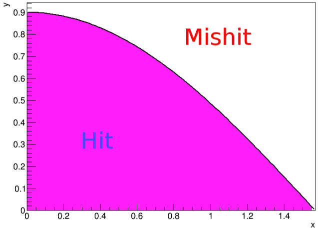
$\Rightarrow$

$$P = E[p(x)] = \int_0^{\pi} p(x)\rho(x)dx = \frac{2l}{\pi L}$$

$$\hat{P}\hat{P} = \frac{n}{N} \xrightarrow{N\to\infty} P = \frac{2l}{\pi L} \Rightarrow \hat{\pi} = \frac{2Nl}{nL}$$

$p(x) 0 < x < \frac{\pi}{2}$

$\Rightarrow$



$$(x, y) : \mathcal{U}(0, \frac{\pi}{2}) \times \mathcal{U}(0, 1) \text{and}$$

$$y \begin{cases} \leq p(x) :, \\ > p(x) :. \end{cases}$$

$w(x, y) = \Theta(p(x) - y)$

$\Theta(x)$

$\rightarrowtail \varrho(x, y) = \rho(x) g(y) = \frac{2}{\pi} \cdot 1$

$\Rightarrow$

$$P = E(w) = \int w(x, y) \varrho(x, y) dx dy = \frac{2l}{\pi L} \xleftarrow{N \to \infty} \hat{P} = \frac{1}{N} \sum_{i=1}^{N} w(x_i, y_i) = \frac{n}{N}$$

$\hat{P}\hat{\sigma} = \frac{1}{\ldots} \sqrt{\frac{n}{\ldots}\left(1 - \frac{n}{\ldots}\right)}$

Lecture1/Headstails

$\Rrightarrow \pi$

$\hookrightarrow (y)(-L, L)\mathcal{U}(-L, L)$

$> LL$

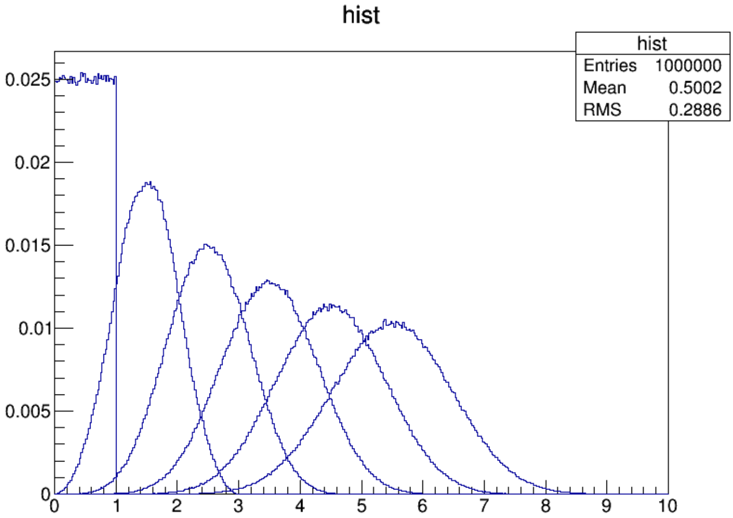
$\hookrightarrow (\phi)(0, \pi) \hookrightarrow y$

$$y_{\max} = y + |\cos\phi| l$$
$$y_{\min} = y - |\cos\phi| l$$

$\hookrightarrow y = L y = 0 y = -L$

N	$\hat{\pi}$	$\hat{\pi} - \pi$	$\sigma(\hat{\pi})$
	3.12317	−0.01842	0.03047
	3.14707	0.00547	0.00979
	3.13682	−0.00477	0.00307
	3.14096	−0.00063	0.00097

Lecture1/CLT



$\Rrightarrow$

$$\frac{1}{N}\sum_{i=1}^{N} f(x_i) \xrightarrow{N\to\infty} \frac{1}{b-a}\int_a^b f(x)dx = E(f)$$

$\Rrightarrow$

$$\sigma = \frac{1}{\sqrt{N}}\sqrt{\left[E(f^{2}) - E^{2}(f)\right]}$$

$\Rrightarrow$

$$P = \int w(x)\rho(x)dx = \int_0^{\pi/2} (\frac{l}{L}\cos x)\frac{2}{\pi}dx = \frac{2l}{\pi L} \xrightarrow{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} w(x_i)$$

$\Rrightarrow$

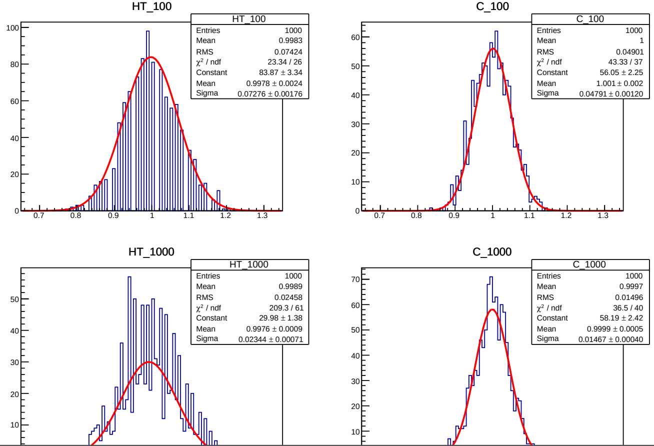
$$\hat{\sigma}^{\text{Crude}} < \hat{\sigma}^{\text{Hitandmishit}}$$

$\Rrightarrow$

Lecture1/CrudevsHT

$\Rrightarrow$

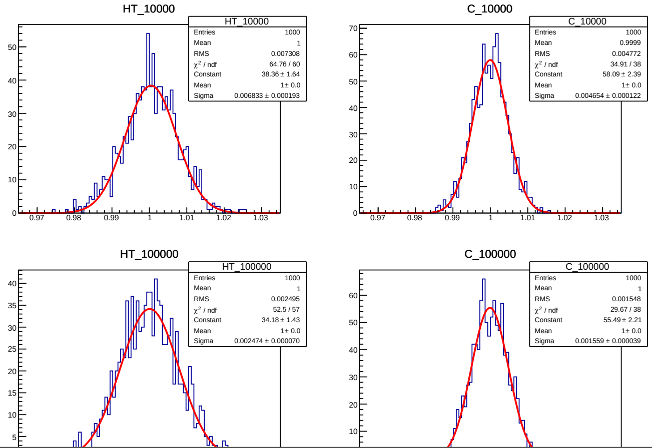
$\hookrightarrow \int_0^{\pi/2} \cos x dx$



Lecture1/CrudevsHT

$\Rrightarrow$

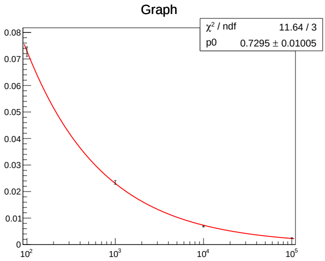
$\hookrightarrow \int_0^{\pi/2} \cos x dx$

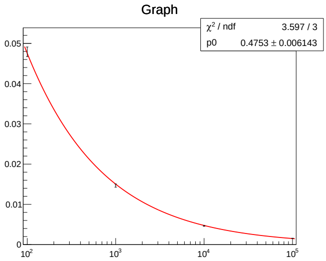


Lecture1/CrudevsHT

$\Rrightarrow$

$\hookrightarrow \int_0^{\pi/2} \cos x dx$





$\Rrightarrow 1/\sqrt{N}$

$\Rrightarrow$

⇛

$$\sigma = \frac{1}{\sqrt{N}}\sqrt{V(f)}$$

⇛

- N

  ⇉

⇛$V(f)\longrightarrow$

⇛

⇛

$$I = \int_0^1 f(u)du = \int_0^a f(u)du + \int_a^1 f(u)du, 0 < a < 1.$$

⇛⇉

j^{th} $w_j n_j w_i n_i$

$$I = \int_{\Omega} f(x)dx, \Omega = \bigcup_{i=1}^{k} w_i$$

j^{th}

$$I_j = \int_{w_j} f(x)dx, \Rightarrow I = \sum_{j=1}^{k} I_i$$

$\Rrightarrow p_j w_j dp_j = \frac{dx}{w_j}$

$\Rrightarrow$

$$\hat{I}_j = \frac{w_j}{n_j} \sum_{i=1}^{n_j} f(x_j^{i})$$

$$\hat{I} = \sum_{j=1}^{k} \hat{I}_j = \sum_{j=1}^{k} \frac{w_j}{n_j} \sum_{i=1}^{n_j} f(x_j^{(i)}),$$

$V(\hat{I}) = \sum_{j=1}^{k} \frac{w_j^{2}}{\,} V_j(f)$ $\hat{V}(\hat{I}) = \sum_{j=1}^{k} \frac{w_j^{2}}{\,} \hat{V}_j(f)$

⇛

⇉

$$f(x)dx \longrightarrow \frac{f(x)}{g(x)}dG(x), \text{where} g(x) = \frac{dG(x)}{dx}$$

- $G(x)\mathcal{U}$
- $w(x) = \frac{f(x)}{g(x)}$
- $\hat{E}(w)\hat{V}_G(w)$

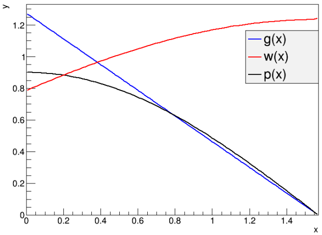
- $g(x)$
-
  - $g(x)$
  - $G(x)g$

$\Rrightarrow \pi$

$\Rrightarrow L = l$

- $g(x) = \frac{4}{\pi}(1 - \frac{2}{\pi}x)$
- $G(x) = \frac{4}{\pi}x(1 - \frac{x}{\pi})$
-

$$w(x) = \frac{p(x)}{g(x)} = \frac{\pi}{4}\frac{\cos x}{1 - 2x/\pi}$$



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$$\sigma_\pi^{\text{IS}} \simeq \frac{0.41}{\sqrt{N}} < \sigma_\pi \simeq \frac{1.52}{\sqrt{N}}$$

-
  -
  -

⇛

$\Rightarrow$

$$\int f(x)dx = \int [f(x) - g(x)]dx + \int g(x)dx$$

- $g(x)$
- $\int [f(x) - g(x)]dx$
- $V(f \to g) \xrightarrow{f \to g} 0$

$\Rightarrow$

-
- $g(x)$

$\Rightarrow$

- $\int g(x)dx$

$\Rrightarrow$

- $f f\prime$
- $V(f + f\prime) = V(f) + V(f\prime) + 2Cov(f, f\prime)$
- $Cov(f, f\prime) < 0$

$\Rrightarrow$

- $f f\prime$

$\Rrightarrow$

-
-
- $f(x)$