Extracting angular observables with Method of Moments

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in collaboration with
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1. Motivation.
2. Method of Moments.
3. Systematic uncertainties.
4. MC toy studies.
5. Conclusions.

Motivation

Likelihood(LL) fits even though widely used suffer from couple of draw backs:

1. In case of small number events LL fits suffer from convergence problems. This behaviour is well known and was observed several times in toys when we done $\mathrm{B} \to \mathrm{K}^{*}\mu\mu$.
2. LL can exhibit a bias when underlying physics model is not well known, incomplete or mismodeled.
3. The LL have problems converging when parameters of the p.d.f. are close to their physical boundaries, so-called "boundary problem"
4. Accessing uncertainty in LL in some cases requires application of computationally expensive Feldman-Cousins method.

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Method of Moments

MoM solves the above problems:

Advantages of MoM

- Probability distribution function rapidity converges towards the Gaussian distribution.
- MoM gives an unbias result even with small data sample.
- Insensitive to large class of remodelling of physics models.
- Is completely insensitive to boundary problems.

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Drawback:

Advantages of MoM

- Estimated uncertainty in MoM is larger then the ones from LL.

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Introduction to MoM

Let us a define a probability density function p.d.f. of a decay:

$$P(\vec{\nu},\vec{\vartheta}) \equiv \sum_i S_i(\vec{\nu}) \times f_i(\vec{\vartheta}) \tag{1}$$

Let's assume further that there exist a dual basis: $\{f_i(\vec{\vartheta})\}$, $\{\tilde{f}_i(\vec{\vartheta})\}$ that the orthogonality relation is valid:

$$\int_\Omega \mathrm{d}\vec{\vartheta}\, \tilde{f}_i(\vec{\vartheta}) f_j(\vec{\vartheta}) = \delta_{ij} \tag{2}$$

Since we want to use MoM to extract angular observables it's normal to work with Legendre polynomials. In this case we can find self-dual basis:

$$\forall_i \tilde{f}_i = f_i \ , \tag{3}$$

just by applying the ansatz: $\tilde{f}_i = \sum_i a_{ij} f_j$.

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Determination of angular observables

Thanks to the orthonormality relation Eq. 2 one can calculate the $S_i(\vec{\nu})$ just by doing the integration:

$$S_i(\vec{\nu}) = \int_\Omega d\vec{\vartheta} P(\vec{\nu}, \vec{\vartheta}) \tilde{f}_i(\vec{\vartheta}) \quad (4)$$

We also need to integrate out the $\vec{\nu}$ dependence:

$$\langle S_i \rangle = \int_\Theta d\vec{\nu} \int_\Omega d\vec{\vartheta} P(\vec{\nu}, \vec{\vartheta}) \tilde{f}_i(\vec{\vartheta}) \quad (5)$$

MoM is basically performing integration in Eq. 5 using MC method:

$$E[S_i] \to \widehat{E[S_i]} = \frac{1}{N} \sum_{k=1}^{N} \tilde{f}(x_k)$$

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Uncertainty estimation

MoM provides also a very fast and easy way of estimating the statistical uncertainty:

$$\sigma(S_i) = \sqrt{\frac{1}{N-1}\sum_{k=1}^{N}(\tilde{f}_i(x_k) - \widehat{S}_i)^2} \tag{6}$$

and the covariance:

$$\mathrm{Cov}[S_i, S_j] = \frac{1}{N-1}\sum_{k=1}^{N}[\widehat{S}_i - \tilde{f}_i(x_k)][\widehat{S}_j - \tilde{f}_j(x_k)] \tag{7}$$

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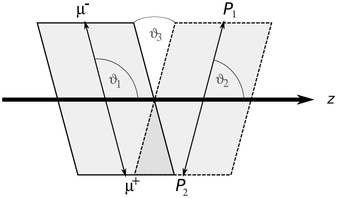
Partial Waves mismodeling

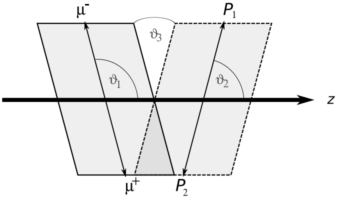
- Let us consider a decay of $B \to P_1 P_2 \mu^- \mu^+$.
- In terms of angular p.d.f. is expressed in terms of partial-wave expansion.
- For $B \to K\pi\mu^-\mu^+$ system, S,P,D waves have been studied.
  - The muon system of this kind of decays has a fixed angular dependence in terms of ϑ_1 and ϑ_3.
  - The hadron system can have arbitrary large angular momentum.



Partial Waves mismodeling

- One can write the p.d.f. separating the hadronic system:

$$P(\cos\vartheta_1, \cos\vartheta_2, \vartheta_3) = \sum_i S_i(\vec{\nu}, \cos\vartheta_2) f_i(\cos\vartheta_1, \vartheta_3) \quad (8)$$



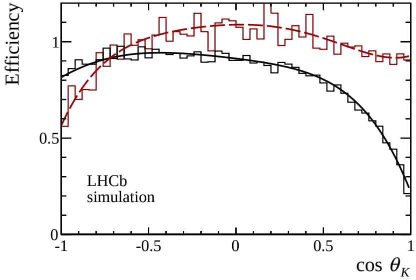
  - $S_i(\vec{\nu}, \cos\vartheta_2)$ can be further expend in terms of Legendre polynomials $p_l^{|m|}(\cos\vartheta_2)$:

$$S_i(\vec{\nu}, \cos\vartheta_2) = \sum_{l=0}^{\text{inf}} S_{k,l}(\vec{\nu}) p_l^{|m|}(\cos\vartheta_2) \quad (9)$$

  - Experimentally the $S_{k,l}$ are easily accessible, but there is a theoretical difficulty as one would need to sum over infinite number of partial waves.

Detector effects

- Since our detectors are not a perfect devices the angular distribution observed by them are not the distributions that the physics model creates.



- To take into account the acceptance effects one needs to simulate the a large sample of MC events.
  Try to figure out the efficiency function.
- Try to figure out the efficiency function.
- Number of possibilities.
- Then you can just weight events:

$$\widehat{E[S_i]} = \frac{1}{\sum_{k=1}^{N} w_k} \sum_{k=1}^{N} w_k \tilde{f}(x_k), \;\; w_k = \frac{1}{\epsilon(x_k)}$$

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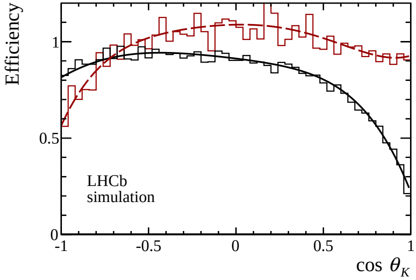

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Unfolding matrix

In general one can write the distribution of events after the detector effects:

$$P^{\rm Det}(x_d) = N \int \int dx_t \ dx_d \ P^{\rm Phys}(x_t) E(x_d|x_t), \qquad (10)$$

where $N^{-1} = \int \int dx_t \ dx_d \ P^{\rm Phys}(x_t) E(x_d|x_t)$ and $(x_d|x_t)$ denotes the efficiency $\epsilon(x_t)$ and resolution of the detector $R(x_d|x_t)$:

$$E(x_d|x_t) = \epsilon(x_t) R(x_d|x_t) \qquad (11)$$

One can define the raw moments:

$$Q_i^{(m)} = \int \int dx_t \ dx_d \ \tilde{f}_i(x_d) P^{(m)}(x_t) E(x_d|x_t) \qquad (12)$$

The m index corresponds to simulation sample that has S_0 and S_m observables set to $\frac{1}{2}$ and rest to zero.

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Unfolding matrix

Once again we can use MC estimator:

$$Q_i^{(m)} \to \widehat{Q}_i^{(m)} = \frac{1}{N_t} \sum_i^{N_d} \tilde{f}_i(x_d^{i,m}) \tag{10}$$

Linearity of the integral ensures that there has to exists a linear transformation:

$$\vec{Q} = M\vec{S}, \tag{11}$$

where M is so-called unfolding matrix, given by the formula:

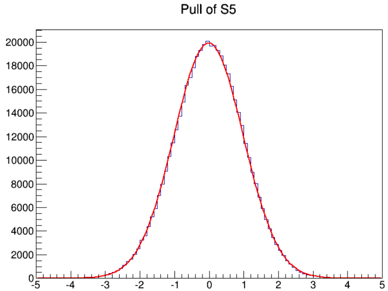
$$M_{ij} = \begin{cases} 2Q_i^{(0)} & j = 0\,, \\ 2\left(Q_i^{(j)} - Q_i^{(0)}\right) & j \neq 0\,, \end{cases} \tag{12}$$

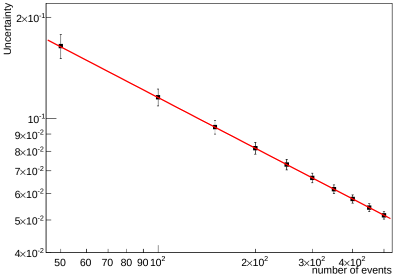
Once we measured the moments Q in data we can invert Eq. 11 and get the $\vec{S}$:

$$\widehat{\vec{S}} = M^{-1}\widehat{\vec{Q}}, \tag{13}$$

Toy Validation

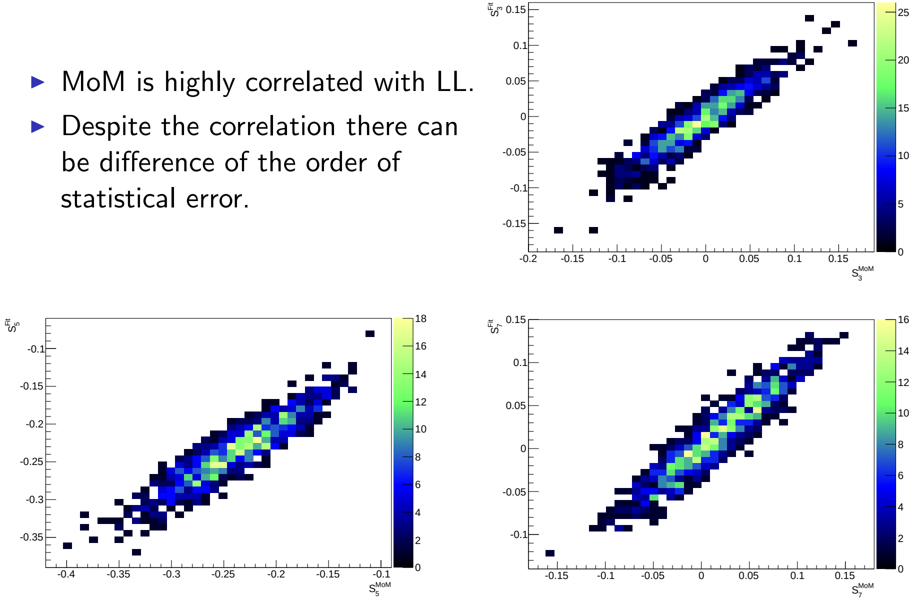
- All the statistics properties of MoM have been tested in numbers of TOY MC.
- As long as you have ~ 30 events your pulls are perfectly gaussian.
- Uncertainty scales with $\frac{\alpha}{\sqrt{n}}$, $\alpha = \mathcal{O}(1)$.
- Never observed any boundary problems.





Correlation of MoM with Likelihood

- MoM is highly correlated with LL.
- Despite the correlation there can be difference of the order of statistical error.



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Conclusions

1. MoM posses several big advantages with one drawback which is larger statistical uncertainty.
2. Allows us to go smaller q^2 bins (get ready for 1 GeV^2 soon!).
3. Alternative method of extracting the detector effects.
4. Can be applied to various rare decays.